

Feynman Propagator and Contour Integrals

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- 4 The contours of the ω dependence term of the Feynman propagator

The Transition Amplitude

Conceptually, you can write down the transition amplitude for NR QM.

- The so-called overlap Integral in QM:

$$= \langle \underline{x}, t | \underline{x}', t' \rangle$$

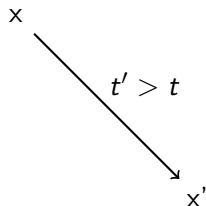
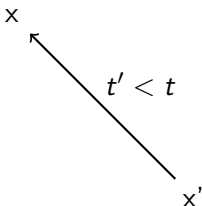
- But, the closest thing we have in QFT resembling position eigenstates:

$$|x'\rangle \equiv \phi(x') |0\rangle$$

$$\langle x| \equiv \langle 0| \phi(x)$$

- So, considering the transition from a localized state $\phi(x) \rightarrow \phi(x')$:

$$= \langle 0 | \phi(x) \phi(x') | 0 \rangle$$



Correlation Functions

Now note, we are dealing with a free field operator $\phi(x)$ with time t and $\phi(x')$ with time t' .

Since time ordering is considered, we can construct two unique correlation functions.

$$\begin{aligned}G_+ &= \langle 0 | \phi(x) \phi(x') | 0 \rangle & (t > t') \\G_- &= \langle 0 | \phi(x') \phi(x) | 0 \rangle & (t < t')\end{aligned}$$

Since $\phi(x)$ and $\phi(x')$ don't commute, we consider a superposition of these states in the commutation relation.

- Of time ordering ($t > t'$)

$$G_R(x, x') = \theta(t - t') \langle 0 | [\phi(x), \phi(x')] | 0 \rangle$$

Correlation Functions

- Of time ordering ($t' > t$)

$$G_A(x, x') = \theta(t' - t) \langle 0 | [\phi(x), \phi(x')] | 0 \rangle$$

And, what I will use for the rest of the talk, the Feynman correlation function.

- Of time ordering ($t' > t$) and ($t > t'$)

$$G_F(x, x') = \theta(t - t') \langle 0 | \phi(x) \phi(x') | 0 \rangle + \theta(t' - t) \langle 0 | \phi(x') \phi(x) | 0 \rangle$$

$$G_F(x, x') = \theta(t - t') G_+ + \theta(t' - t) G_-$$

Feynman Propagator

Starting with the Feynman correlation function, G_F we know that the free field operator is

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega}} [a(\underline{k}) e^{ikx} + a^\dagger(\underline{k}) e^{-ikx}]$$

$$\phi(x') = \int \frac{d^3 k'}{(2\pi)^3 \sqrt{2\omega}} [a(\underline{k}') e^{ik'x'} + a^\dagger(\underline{k}') e^{-ik'x'}]$$

Recall $G_F(x, x') = \theta(t - t') G_+ + \theta(t' - t) G_-$

$$\begin{aligned} G_+ &= \langle 0 | \phi(x) \phi(x') | 0 \rangle \quad (t > t') \\ &= \langle 0 | \int \tilde{d}k [a(\underline{k}) e^{ikx} + a^\dagger(\underline{k}) e^{-ikx}] \int \tilde{d}k' [a(\underline{k}') e^{ik'x'} + a^\dagger(\underline{k}') e^{-ik'x'}] | 0 \rangle \end{aligned}$$

Feynman Propagator (continued)

$$\begin{aligned} &= \int \tilde{d}k \tilde{d}k' \langle 0 | [a(\underline{k})e^{ikx} + a^\dagger(\underline{k})e^{-ikx}] [a(\underline{k}')e^{ik'x'} + a^\dagger(\underline{k}')e^{-ik'x'}] | 0 \rangle \\ &= \int \tilde{d}k \tilde{d}k' \langle 0 | [a(\underline{k})e^{ikx}] [a^\dagger(\underline{k}')e^{-ik'x'}] | 0 \rangle \\ &= \int \tilde{d}k \tilde{d}k' \langle 0 | [a(\underline{k})e^{ikx}] [a^\dagger(\underline{k})e^{-ik'x'}] | 0 \rangle \end{aligned}$$

From the operators acting on the vacuum, you get a $\delta^{(3)}(k - k')(2\pi)^3$. This collapses the $\tilde{d}k'$ integral.

$$\begin{aligned} G_+ &= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik(x-x')} \\ \theta(t - t') G_+ &= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik(x-x')} \theta(t - t') \\ \theta(t' - t) G_- &= \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik(x'-x)} \theta(t' - t) \end{aligned}$$

Feynman Propagator (continued)

Now recall that $G_F(x, x') = \theta(t - t')G_+ + \theta(t' - t)G_-$, hence:

$$\begin{aligned}G_F &= \int \tilde{d}k [e^{-i(k(x-x'))}\theta(t - t') + e^{-i(k(x'-x))}\theta(t' - t)] \\&= \int \tilde{d}k e^{-i\underline{k}(\underline{x}-\underline{x}')} [e^{i\omega(t-t')}\theta(t - t') + e^{-i\omega(t-t')}\theta(t' - t)] \\&= \int \tilde{d}k e^{-i\underline{k}(\underline{x}-\underline{x}')} [e^{i\omega\tau}\theta(\tau) + e^{-i\omega\tau}\theta(-\tau)], \quad \text{where } \tau = t - t' \\&= \int \frac{d^3k}{(2\pi)^4} \frac{1}{2\omega} e^{-i\underline{k}(\underline{x}-\underline{x}')} \lim_{\epsilon \rightarrow 0} \left[\int_{-\infty}^{+\infty} \frac{i d\omega_I}{\omega_I^2 - \omega^2 + i\epsilon} e^{i\omega_I(t-t')} \right], \quad (\text{Schwartz Eq. 6.28}). \\&= \int \frac{d^3k}{(2\pi)^3} e^{-i\underline{k}(\underline{x}-\underline{x}')} G(t - t')\end{aligned}$$

$G(t - t')$ is a complex integral, and our job is to evaluate it!

Cauchy's Integral Formula

Now, let's do some Complex Analysis. Suppose, for $f(z)$ analytic, of the form

$$\oint f(z) = \oint \left(\frac{R_i(z)}{z - z_i} + \dots \right)$$

Integrals of such form should have i number of poles at $z_i = z$ alongside i residuals $R_i(z)$. An integral of one pole evaluates to:

$$\oint f(z) = \oint \left(\frac{R_i(z)}{z - z_0} \right) = 2\pi i R_0(z_0)$$

Likewise, an integral of i poles gives us:

$$\oint f(z) = \oint \left(\frac{R_i(z)}{z - z_i} + \dots \right) = 2i\pi \sum_i R_i(z_i)$$

Example: Evaluate the following

$$G = \oint \frac{dz}{4z^2 - 1}$$

We need to get it in the right form. Do partial fractions.

$$G = \frac{1}{4} \oint dz \left(\frac{1}{z - 1/2} - \frac{1}{z + 1/2} \right)$$

Our Residues evaluated at $z_0 = 1/2$ and $z_1 = -1/2$:

$$R_0 = 1 \text{ and } R_1 = -1$$

hence,

$$2i\pi \sum_i R_i(z_i) = 2i\pi(R_0 + R_1) = 2i\pi(0) = 0$$

Nice! Let's apply this method to $G(t - t')$, from Srednicki

Evaluating Complex Integral of the Propagator G_F

Let's recall the $G(t - t')$ term from the Feynman Propagator

$$G(t - t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega_l \frac{i}{\omega_l^2 - \omega^2 + i\epsilon} e^{i\omega_l(t-t')}$$

What we want is a sum of terms over singularities (or poles in our case). We need to do partial fraction decomposition to get there.

$$\begin{aligned} &= \frac{i}{\omega_l^2 - \omega^2 + i\epsilon} \\ &\equiv \frac{i}{[\omega_l - (\omega - i\epsilon)][\omega_l - (-\omega + i\epsilon)]} \\ &\equiv \frac{i}{[\omega_l - (\omega - i\epsilon)]} + \frac{i}{[\omega_l - (-\omega + i\epsilon)]} \end{aligned}$$

Hence,

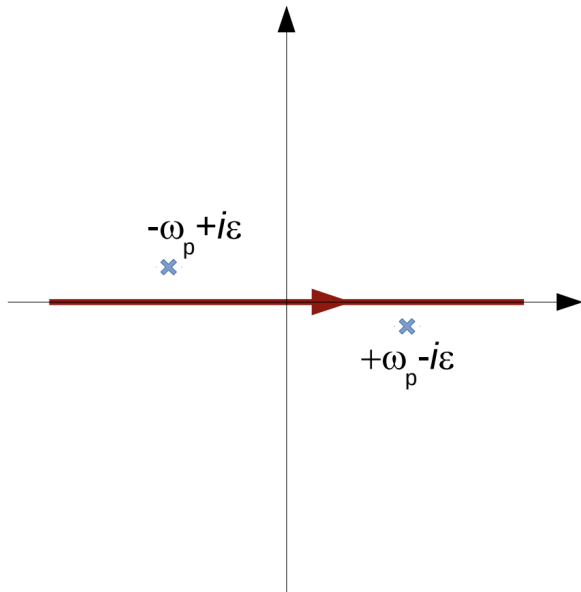
$$G(t - t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega_I \left(\frac{i}{[\omega_I - (\omega - i\epsilon)]} + \frac{i}{[\omega_I - (-\omega + i\epsilon)]} \right) e^{i\omega_I(t-t')}$$

Now, let $\omega_I \rightarrow z$

$$G(t - t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \left(\frac{i}{[z - (\omega - i\epsilon)]} + \frac{i}{[z - (-\omega + i\epsilon)]} \right) e^{iz(t-t')}$$

Notice! We have two poles, or singularities to be more precise. We have them at:
 $z = \omega - i\epsilon$ and $z = -\omega + i\epsilon$

The two poles

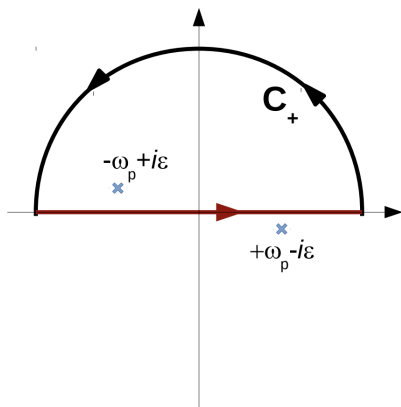
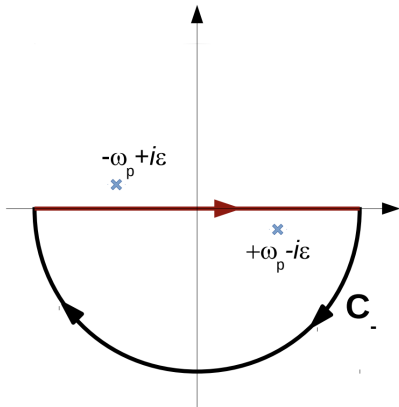


Ok! Given that I have two poles, and two terms of my integral, I can pick my bounds in which I'll carry out the contour integral.

$\omega_I \rightarrow z$

$$= \frac{1}{2\pi} \oint_{C_1} dz \left(\frac{i}{[z - (\omega - i\epsilon)]} \right) e^{iz(t-t')} + \frac{1}{2\pi} \oint_{C_2} dz \left(\frac{i}{[z - (-\omega + i\epsilon)]} \right) e^{iz(t-t')}$$

Since first term is in upper half, C_1 is a contour boundary above. Likewise, C_2 bounds are below.



Final computation

Let us now carry out the Cauchy integral with the residue at the poles.

$$= \frac{1}{2\pi} \frac{1}{2\omega} \oint_{C_1} dz \left(\frac{ie^{iz(t-t')}}{[z - (\omega - i\epsilon)]} \right) + \frac{1}{2\pi} \frac{1}{2\omega} \oint_{C_2} dz \left(\frac{ie^{iz(t-t')}}{[z - (-\omega + i\epsilon)]} \right)$$

We have two Residues $R_0 = ie^{iz_0(t-t')}$ and $R_1 = ie^{iz_1(t-t')}$. And,
 $z_0 = (\omega - i\epsilon)$ $z_1 = (-\omega + i\epsilon)$

$$R_0 = \lim_{\epsilon \rightarrow 0} \left(ie^{(\omega - i\epsilon)(t-t')} \right) = ie^{i\omega(t-t')}$$

$$R_1 = \lim_{\epsilon \rightarrow 0} \left(ie^{(-\omega + i\epsilon)(t-t')} \right) = ie^{-i\omega(t-t')}$$

$$\begin{aligned} \frac{1}{2\pi} \frac{1}{2\omega} \oint f(z) &= \oint \left(\frac{R_i(z)}{z - z_i} + \dots \right) = \frac{1}{2\pi} \frac{1}{2\omega} 2\pi (R_0(z_0) + R_1(z_1)) \\ &= \frac{1}{2\omega} i (e^{i\omega(t-t')} + e^{-i\omega(t-t')}) \end{aligned}$$

$$= \frac{i}{2\omega} (e^{-i\omega|t-t'|})$$

QED

The End

References

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- Arfken, G. B., Weber, H. J., Harris, F. E., *Mathematical Methods for Physicists*. 7th Edition, Elsevier, 2012. (For the Cauchy Integral).